THE FOURTEENTH WESTERN MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE fourteenth regular western meeting of the American Mathematical Society, being the forty-fifth regular meeting of the Chicago Section, was held at the University of Chicago on Friday and Saturday, April 9 and 10, 1920. The total attendance at this meeting was one of the largest in the history of the Chicago Section, there being about one hundred and fifty persons present at the meeting of Friday afternoon. Among these were the following sixty-nine members of the Society:

Professor G. A. Bliss, Professor H. Blumberg, Professor R. L. Borger, Professor W. C. Brenke, Professor E. W. Brown, Professor W. D. Cairns, Dr. C. C. Camp, Dr. J. W. Campbell, Professor J. A. Caparo, Professor R. D. Carmichael, Professor A. B. Coble, Professor A. R. Crathorne, Mr. H. W. Curjel, Dr. H. B. Curtis, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor Arnold Emch, Mr. E. B. Escott, Mr. T. C. Fry, Professor M. E. Graber, Professor Harris Hancock, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor G. O. James, Professor A. M. Kenyon, Dr. J. R. Kline, Professor W. C. Krathwohl, Professor J. W. Lasley, Jr., Professor Kurt Laves, Professor A. C. Lunn, Professor J. V. McKelvey, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor Max Mason, Professor T. E. Mason, Professor Bessie I. Miller, Professor W. L. Miser, Professor E. H. Moore, Professor E. E. Moots, Professor E. J. Moulton, Professor F. R. Moulton, Professor G. W. Myers, Professor C. I. Palmer, Professor A. D. Pitcher, Professor S. E. Rasor, Professor P. R. Rider, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Professor D. A. Rothrock, Professor Oscar Schmiedel, Professor G. T. Sellew, Professor E. B. Skinner, Professor H. E. Slaught, Professor R. B. Stone, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor K. P. Williams, Professor R. E. Wilson, Professor C. H. Yeaton, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

On Friday evening seventy-four members and guests gathered at a dinner at the Quadrangle Club.

The session of Friday afternoon, presided over by Professors E. W. Brown and D. R. Curtiss, was devoted to a symposium on "The Maxwell field equations and the theory of relativity." Papers were read by Professors Max Mason and A. C. Lunn; the discussion was participated in by Professors E. W. Brown and Alexander Ziwet. Synopses of the symposium papers follow:

I. "Electromagnetic Field Equations," by Professor MAX MASON.

1. Definition of the vectors E, B. The field equations for charge and free space, and the retarded potentials.

2. The Lorentz electron hypothesis. The electromagnetic equations for ponderable bodies; actual and averaged value of the vectors.

3. Comparison with the action at a distance theories (Weber, Clausius et al.) and with the action in the medium theories (Maxwell, Hertz).

4. The equation of activity and of energy. The dynamics of the electron.

5. Types of analytical problems in field determinations.

6. Ponderable matter and free electrons; thermionic space charge and current.

References.

Whittaker, History of the Theories of Aether and Electricity.
Encyclopädie der mathematischen Wissenschaften, V. 12 (Reiff and Sommerfeld), V. 13, 14 (Lorentz).
Lorentz, The Theory of Electrons.

II. "The Theory of Relativity," by Professor A. C. LUNN.

1. Terrestrial mechanics and geometric optics on a laboratory scale. The experimental aspect of euclidean metrics, optical meaning of straightness. The materialization of space by the ideal rigid body. Practical chronometers. The group of transformations of newtonian mechanics. The gravitational field. Large scale geodesy, with possible non-euclidean phrasing.

2. Celestial mechanics. Extension of application of the general laws of mechanics and success of the law of gravitation when observations are reduced on the basis of euclidean triangulation and rectilinear path of light. Exception in

the case of the line of apses of Mercury. Determination of inertial systems. The "absoluteness" of rotation.

3. Optics and ether theories. Development of the wave-theory of light in its successive forms. Experimental basis of the notion of fixed ether. Problem of optical determination of absolute motion and negative outcome of the Michelson-Morley experiment. The Fitz-Gerald contraction as a presumptive physical phenomenon.

4. Electromagnetic theory and electrons. Distinct physical effects of ether displacement and polarization currents. The Larmor-Lorentz equations and their group of transformations. Comparison with newtonian relativity.

5. Einstein's first or special form of relativity in terms of the constancy of the velocity of light and the mutual dependence of space and time measurements. Minkowski's theory of euclidean four-dimensional space-time and the corresponding vector analysis or algebraic covariance theory associated with the invariant sum of squares. Real interpretations of non-euclidean type. The tensor of stress, momentum and density.

6. General relativity in terms of covariant theory of general quadratic differential form in four variables. The "local" affine vector analysis of an infinitesimal region associated with the group of all linear transformations. Curvature of finite regions of space-time and application of covariant differentiation to the statement of physical relations. The Riemann-Christoffel four-index tensor and the associated reduced tensor.

7. The general dynamical tensor and Einstein's law of gravitation. Newton's law as a first approximation. Equations of motion for particles in the void. Path of a ray of light. Explanation of the residual in the case of Mercury and prediction of bending of ray of light passing near the sun. Outlook on other phenomena. Relation to molecular theories.

References.

Einstein: *Annalen der Physik*, volume 17 (1905), pages 891–921 and volume 49 (1916), pages 769–822; *Jahrbuch der Radioaktivität*, volume 4 (1907), pages 411–462.

Laub and Ishiwara: *Jahrbuch der Radioaktivität*, volume 7 (1910), pages 405–463 and volume 9 (1912), pages 560–648.

Minkowski: *Göttinger Nachrichten*, 1908, pages 53–111, and *Mathematische Annalen*, volume 68 (1910), pages 472–525.

Eddington: Report on the Relativity Theory of Gravitation. London, Fleetway Press, 1918.

At the sessions of Friday and Saturday forenoons, presided over by Professor Carmichael, chairman of the section, and Professor Wilczynski respectively, the following papers were read:

(1) Professor G. A. MILLER: "Transitive constituents of the group of isomorphisms of an abelian group of order p^m."

(2) Professor G. E. WAHLIN: "On the unessential divisors of the discriminants of a domain."

(3) Professor E. B. STOUFFER: "Seminvariants of a general system of linear homogeneous differential equations."

(4) Professor E. J. WILCZYNSKI: "A set of characteristic properties of a class of congruences connected with the theory of functions."

(5) Professor A. B. COBLE: "Porisms."

(6) Professor O. E. GLENN: "Modular covariants of formal type."

(7) Professor T. E. MASON: "New pairs of amicable numbers."

(8) Professor DUNHAM JACKSON: "On functions of closest approximation."

(9) Professor R. D. CARMICHAEL: "On the expansion of certain analytic functions in series."

(10) Professor R. D. CARMICHAEL: "Boundary value and expansion problems: Algebraic basis of the theory" (preliminary communication).

(11) Professor R. D. CARMICHAEL: "Boundary value and expansion problems: Formulation of various transcendental expansion problems" (preliminary communication).

(12) Professor D. R. CURTISS: "On Jensen's extension of Rolle's theorem."

(13) Professor K. P. WILLIAMS: "The spiral descent of an airplane."

(14) Dr. GEORGE RUTLEDGE: "An invariant area property of polynomial curves of odd degree and the direct evaluation of Cotes's coefficients."

(15) Professor P. R. RIDER: "The minimum area between a curve and its caustic."

(16) Professor W. D. MACMILLAN: "On the moment of inertia in the problem of n bodies."

(17) Dr. J. R. KLINE: "Concerning the boundary of uniformly connected 'im kleinen' domains."

(18) Professor E. H. MOORE: "On the reciprocal of the general algebraic matrix."

(19) Professor W. H. ROEVER: "Determination of the wind structure from the flight of a pilot balloon."

The papers of Professors Miller, Wahlin, Stouffer, Glenn, and of Dr. Rutledge were read by title. Abstracts of the papers, numbered in accordance with the above list, are given below:

1. An abelian group cannot contain more than two characteristic operators. Hence the group of isomorphisms I of an abelian group G, when represented on letters corresponding to non-characteristic operators of G, cannot contain more than two transitive constituents which are separately simply isomorphic with I and a necessary and sufficient condition that it contains two such constituents is that the order of G is twice an odd number. When the order of G is p^m, p being a prime number, then the I of G contains only one transitive constituent which is simply isomorphic with G. Professor Miller investigates the relations existing between the various constituents of I, especially those corresponding to operators of the same order contained in G. In particular, he proves that to the identity of one such constituent there corresponds an abelian group of order p^a and of type $(1, 1, 1, \cdots)$ in the next higher constituent, and he determines the orders of these groups.

2. Various investigators have made extensive studies of the unessential divisors of the discriminants of the integers of an algebraic domain. In his Theorie der algebraischen Zahlen, Hensel shows that if p is a common unessential divisor of the discriminants of all integers of a domain and if in the domain all the prime factors of p are of the first degree, then p is necessarily less than the degree of the domain. In the case when all the prime factors of p are not of the first degree, he finds a larger upper limit for the primes which may be common unessential divisors. In the *Mathematische Annalen*, volume 73, von Zylinski shows that p is always less than n, if p is a common unessential discriminant divisor. In the present paper, Professor Wahlin shows that in case all the prime divisors of p are of higher degree than the first, a still smaller limit can be fixed.

3. Consider the system of linear homogeneous differential equations

$$y_i^{(m)} + \sum_{l=0}^{m-1} \sum_{k=1}^{n} \binom{m}{l} p_{ikl} y_k^{(l)} = 0 \qquad (i = 1, 2, \cdots, n),$$

where p_{ikl} are functions of the independent variable x. It is known that the most general transformation of the dependent variables which leaves this system unchanged in form is given by the equations $y_k = \sum_{\lambda=1}^{n} \alpha_{k\lambda}(x)\bar{y}_\lambda$, $|\alpha_{k\lambda}| \neq 0$, where α_{kl} are arbitrary functions of x. A function of the coefficients and their derivatives which has the same value for the original system as for the transformed system is called a seminvariant. Professor Stouffer calculates a complete system of seminvariants for the above set of equations. The seminvariants for the simpler case when $p_{ikm-1} = 0$, were first obtained, and the seminvariants for the general case were obtained from them by simple transformations.

4. If two complex variables, z and w, are transferred to the same sphere, and if the points which correspond to each other in a functional relation between z and w are joined by straight lines, the lines thus obtained form a congruence. The general properties of such congruences were determined by Professor Wilczynski in a paper in the *Transactions*, October, 1919. In the present paper, Professor Wilczynski shows that a certain list selected from these properties is characteristic of such congruences. It is a notable fact that most of these properties are of a purely projective character.

5. Professor Coble's paper is concerned with multiple binary forms which have the closure property. A poristic form of this sort defines a complementary poristic form such that the product of the two can be expressed as a determinant which by its very nature is poristic.

6. Professor Glenn's paper presents in brief the main results developed in the author's memoir on formal invariancy, read before the Society in December 1919 (see abstract in this BULLETIN for March, 1920). Certain further developments of general theory are included.

7. By making use of Lehmer's Table of Primes, Professor Mason has been able to show that the numbers $256 \cdot 8520191$

and $256 \cdot 257 \cdot 33023$ are a pair of amicable numbers. This pair of numbers belongs to one of the types treated by Euler, but the large prime was beyond the limits of his table of primes. Other number pairs are obtained by Euler's methods or modifications of his methods.

8. The determination of the best polynomial of approximation for a given continuous function $f(x)$ in a given interval, the degree of the polynomial being given, depends on the meaning attached to the phrase "best approximation." The polynomial for which the maximum of the absolute value of the error is as small as possible is known as the Tchebychef polynomial, and has been extensively studied. The polynomial for which the integral of the square of the error is a minimum is the sum of the first terms of the Legendre's series for $f(x)$, and its properties are also well known. Professor Jackson considers the polynomials for which the integral of the mth power of the error is a minimum, where m is any even positive integer, or, more generally, the integral of the mth power of the absolute value of the error, where m is any number greater than 1. It is found that certain theorems which are well known for $m = 2$ are applicable in the more general case; and it is shown that the approximating polynomial corresponding to exponent m approaches the Tchebychef polynomial as a limit when m becomes infinite. The discussion is put in such a form as to apply also to approximation by finite trigonometric sums, for example, and generally to approximation by linear combinations of a given set of linearly independent functions, having such further properties as to insure the uniqueness of the best approximating function in the sense of Tchebychef.

9. For a certain class $S(x)$ of cases of the $\overline{\Omega}$-series introduced by Professor Carmichael in his paper in the *Transactions*, volume 17, he now treats anew the problem of the representation of functions in such series. The basic function $g(x)$ is restricted to the special asymptotic form

$$g(x) \sim x^{\mu-\rho x} e^{\alpha+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right),$$

valid in a suitable sector V. The object of the present note is to derive certain necessary and sufficient conditions for the

representation of functions in the form of series $S(x + a)$, where a is a suitable constant depending on the function to be represented.

10. Lying back of several well-known expansion problems in the theory of integral equations and of linear differential equations, both ordinary and partial, is a corresponding problem for linear algebraic equations of which the transcendental problems are limiting cases, each limiting case being realized in a way peculiar to its own class. The object of Professor Carmichael's second paper is to study systematically and to generalize in several directions the algebraic problems which thus come into the foreground, the primary purpose being to come to such understanding of these algebraic matters as will make them serve in the fullest way possible as a guide to the formulation and solution of important transcendental problems of expansion.

11. With the results of his second paper as a guide, Professor Carmichael formulates in his third paper several transcendental expansion problems (in association with certain boundary conditions when these are relevant) having to do with equations and systems of equations of the following classes: linear differential equations, both ordinary and partial; linear difference equations, both ordinary and partial; integral equations; integro-differential equations; integro-difference equations; differentio-difference equations, and some others. The systems mentioned may be made up of equations of only one class or of equations from two or more different classes. This preliminary communication announces the formulation of these problems and a certain small progress toward their solution.

12. The theorem of Jensen according to which the complex roots of the derived functions of a polynomial with real coefficients must lie within ellipses of a certain system gives little further information regarding their location. In the present paper, Professor Curtiss obtains results of a more precise nature, and applies these to the problem of locating the roots of the polynomial when the roots of a derivative are known.

13. The orientation of an airplane when making a spiral descent is determined by two angles. One of these angles is given at once in terms of the angle of attack of the machine,

and the other is obtained from the root of a cubic equation given by Devillers. Professor Williams gives a graphical solution of this cubic equation, and also the explicit expansion of the root in question by means of the formula of Lagrange.

14. Newton, in 1711, and Cotes, in 1722, used polynomial curves for approximating definite integrals. Cotes computed the coefficients which bear his name for polynomials up to degree ten. In the present paper, Dr. Rutledge evaluates, by means of determinants, Cotes's coefficients for the general case of even order. A fact brought out by this evaluation is that the area from $x = a$ to $x = b$ under any polynomial curve of degree $2n + 1$ through $2n + 1$ points equally spaced as to abscissas is an invariant of the set of curves of degree $2n + 1$ through these points.

15. Professor Rider's paper considers the problem of joining two fixed points by a curve which with its caustic and the reflected rays at the fixed points will enclose a minimum area. Euler proposed and solved a similar problem concerning a curve and its evolute. For the evolute problem the minimizing curve is a cycloid. For the caustic problem it is a transcendental curve of more complex type. Its equations are obtained in parametric form. The determination of the arbitrary constants in the solution is considered. Finally, a solution is given of a more general problem which includes the evolute and caustic problems as special cases.

16. For the problem of n bodies, Eddington has given the differential equation for the moment of inertia

$$1/2 \cdot d^2I/dt^2 = 2T + W,$$

where T is the kinetic energy and $W = -k^2\Sigma\Sigma m_i m_j/r_{ij}$ is the potential energy, so that $T + W = E$ is the total energy. If E is positive it is evident that I'' is always positive and ultimate dispersion is inevitable. In a steady state of motion, $2T + W = 0$, as was remarked by Poincaré. Professor MacMillan supposes that a system of bodies maintains its configuration in undergoing a process of expansion and contraction. Under this hypothesis, W is inversely proportional to the square root of I and the equation for I may be written $1/2 \cdot d^2I/dt = 2E + k_1^2/\sqrt{I}$. The integrals of this equation show that if E is negative the period of the oscillation is

$P = k_1^2/(-2E)^{3/2}$, which is independent of the amplitude of the oscillation. If the amplitude of the oscillation be increased to the point where the system collapses ($I = 0$), then the maximum value of $\sqrt{I}$ is twice the value which it has for the steady state in which the amplitude is zero. In other words, the system could not expand to double its size for a steady state without eventually collapsing. Applications are made to star clusters.

17. Dr. Kline shows that if R is a uniformly connected "im kleinen," bounded, two-dimensional domain whose exterior is also uniformly connected "im kleinen," then there exists a finitely connected domain Q whose boundary consists of a finite number of simple closed curves, no two of which have a point in common, such that $Q - R$ is a totally disconnected set. Using this result, he shows that necessary and sufficient conditions that a bounded two-dimensional domain R be finitely connected and bounded by a finite number of simple closed curves, no two of which have a point in common, are (1) that R be fully expanded, and (2) that the exterior of R be uniformly connected "im kleinen." The notion of a set being uniformly connected "im kleinen" on every convergent sequence belonging to the set is then defined. It is shown that any bounded set M, that is uniformly connected "im kleinen" on every convergent sequence belonging to M, is uniformly connected "im kleinen." This is not necessarily true, however, if M is unbounded. Using this notion, it is shown that if R is a proper subset of a two-dimensional euclidean space which is unbounded and simply connected, then a necessary and sufficient condition that R have as its boundary an open curve is that R be uniformly connected "im kleinen" on every convergent sequence.

18. In this paper Professor Moore calls attention to a useful extension of the classical notion of the reciprocal of a non-singular square matrix. Consider any $m \times n$ matrix κ, i.e., an array in m rows and n columns of mn complex numbers $\kappa(uv)$ $(u = 1, \cdots, m;\ v = 1, \cdots, n)$. There exists one and only one $n \times m$ matrix λ, *the reciprocal of* κ, such that (1) the columns of λ are linear combinations of the conjugates of the rows of κ, (2) the rows of λ are linear combinations of the conjugates of the columns of κ, (3) the matrix $TS\kappa\lambda\kappa$ obtained by matricial composition of the matrices κ, λ, κ is the original

matrix κ: $TS\kappa\lambda\kappa = \kappa$, viz., $\kappa(uv) = \Sigma_{ts}\, \kappa(ut)\lambda(ts)\kappa(sv)$ (uv), where u, s run from 1 to m, and v, t run from 1 to n while S; T denote summations over the respective ranges $1, \cdots, m$; $1, \cdots, n$. If κ is of rank r, then λ is given explicitly as follows:

$$(r \geqq 2)\ \lambda(v_1u_1) = \sum_{\substack{u_2 < \cdots < u_r \\ v_2 < \cdots < v_r}} \kappa\binom{u_2 \cdots u_r}{v_2 \cdots v_r}\bar{\kappa}\binom{u_1u_2 \cdots u_r}{v_1v_2 \cdots v_r} \div \sum_{\substack{s_1 < \cdots < s_r \\ t_1 < \cdots < t_r}} \kappa\binom{s_1 \cdots s_r}{t_1 \cdots t_r}\bar{\kappa}\binom{s_1 \cdots s_r}{t_1 \cdots t_r} \quad (u_1v_1);$$

$$(r = 1)\ \lambda(vu) = \bar{\kappa}(uv) \div \sum_{st} \kappa(st)\bar{\kappa}(st) \quad (uv);$$

$$(r = 0)\ \lambda(vu) = 0 \quad (uv);$$

where as usual $\kappa\binom{g_1 \cdots g_k}{h_1 \cdots h_k}$ denotes the determinant of the k^2 numbers $\kappa(g_ih_j)$ $(i, j = 1, \cdots, k)$ and $\bar{\kappa}\binom{g_1 \cdots g_k}{h_1 \cdots h_k}$ denotes the determinant of the conjugate numbers $\kappa(g_ih_i)$.

The relation between κ and λ is mutual: κ is the reciprocal of λ, viz., (4), (5): the columns (rows) of κ are linear combinations of the conjugates of the rows (columns) of λ; (6) $ST\lambda\kappa\lambda = \lambda$. The linear combinations of the columns of $\kappa(\lambda)$ are the linear combinations of the conjugates of the rows of $\lambda(\kappa)$ and constitute the m-ary vectors μ (the n-ary vectors ν) of a linear r-space M (N) lying in the complete m-space (n-space) of all m-ary (n-ary) vectors. Let $\bar{M}$ $(\bar{N})$ denote the conjugate space of the conjugate vectors $\bar{\mu}$ $(\bar{\nu})$. Then the matrices $\kappa\,\lambda$ establish $1-1$ linear vector correspondences between the spaces M, $\bar{M}$ and the respective spaces N, $\bar{N}$; $\mu = T\kappa\nu$ is equivalent to $\nu = S\lambda\mu$ and $\bar{\nu} = S\bar{\mu}\kappa$ is equivalent to $\bar{\mu} = T\bar{\nu}\lambda$.

19. Professor Roever shows how, under certain assumptions, it is possible to determine from a single pilot balloon flight the nature of the vector field which represents the wind structure. For this purpose the wind chart and the differential equations of motion of the balloon are needed. These differential equations are obtained from those of a projectile by substituting for the weight of the projectile the lift of the balloon. The wind chart furnishes the finite equations of motion of the balloon. From these two sets of equations the form of the vector field along the path of the balloon is determined, and, under the given assumptions, this is sufficient to

determine the nature of the vector field representing the wind structure.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION

THE thirty-fifth regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 10. The chairman, Professor Blichfeldt, presided. The total attendance was twenty-three, including the following thirteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor H. F. Blichfeldt, Professor Thomas Buck, Professor Florian Cajori, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Dr. Pauline Sperry.

The action of the executive committee in making plans for a meeting of the Section at the University of Washington June 17–19, in connection with the Pacific division of the American association for the advancement of science was approved. It was decided that this should be a special meeting of the Section, the regular Fall meeting to take place October 23, 1920, as scheduled.

The following papers were presented and discussed:

(1) Professor H. F. BLICHFELDT: "On the approximate representation of irrational numbers" (preliminary report).

(2) Professor FLORIAN CAJORI: "Note on the history of divergent series."

(3) Professor L. M. HOSKINS: "Note on the Lorentz transformation and the notion of time."

(4) Professor E. T. BELL: "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind."

(5) Professor E. T. BELL: "Certain remarkable sums related to 3, 5, and 7 squares."

(6) Professor E. T. BELL: "Parametric solutions for a fundamental equation in the general theory of relativity."

In the absence of the author, Professor Bell's papers were read by title. Abstracts of the papers follow below.

1. The theory of approximate solutions, in terms of integers, of linear equations with irrational coefficients, has been developed mainly by Kronecker and Minkowski. Thus, the equations $x_i - a_i x_n = 0$ $(i = 1, 2, \cdots, n-1)$ possess an infinite number of sets of integral solutions with errors $\epsilon_i < \frac{n-1}{n} \cdot \frac{1}{x_n^{1/(n-1)}}$. Kronecker has shown that the equations $x_i - a_i x_n = b_i$ $(i = 1, 2, \cdots, n-1)$ possess an infinite number of sets of integral solutions, with errors ϵ_i at the outset assigned as small as we please, provided no equation exists of the form $\sum_i k_i a_i = k_n$, where the k_i, k_n are integers. Professor Blichfeldt finds that we cannot in general assume $\epsilon_i < N x_n^{-1/(n-1)}$, where N is any given constant.

2. Professor Cajori pointed out that, as early as 1843, A. De Morgan advanced a summation formula for divergent series, discussed asymptotic series and used a principle relating to divergent series; these researches entitle him to the rank of a pioneer in the development of the modern theory of divergent series.

3. Professor Hoskins called attention to an apparent paradox resulting from Einstein's interpretation of the Lorentz transformation. If two exactly similar clocks C and C' keep time correctly for a system S while both are at rest at a point A of the system, and if C remains at A while C' is carried in any closed path with uniform speed v and brought back to A, then according to Einstein the elapsed time as indicated by C' will be less than that indicated by C, the relation being approximately $T' = T[1 - \frac{1}{2}(v/V)^2]$. It is, however, equally legitimate to refer the motion to a coordinate system with respect to which C' remains stationary while C describes a closed path with uniform speed v, and from this point of view the reasoning of Einstein leads to the conclusion that $T = T'[1 - \frac{1}{2}(v/V)^2]$. There seems no ground for regarding the reasoning as valid in one case and not in the other. Is it not possible that in both cases it is based upon an illegitimate physical interpretation of the Lorentz transformation?

4. The twelve elliptic functions considered by Glaisher and other writers are shown in Professor Bell's paper to arise as degenerate forms from a set of sixteen doubly periodic functions of the second kind which may be generated by the operations of a certain group from any one of them. The same functions give also, by differentiation, transformations of the second order, etc., all of the doubly periodic functions of the third kind considered by Hermite, Biehler, Appell, and others, with many more. The paper will appear in the *Messenger of Mathematics*.

5. In connection with his recent determinations of the numbers of representations of integers as sums of 3, 5, 7, 9, 11, and 13 squares, (which will appear in the *American Journal*), Professor Bell remarked that incidentally the 3-square results yield the class number by finite processes only. In this paper new binary class number formulas and analogous results related to quinary and senary forms are derived.

6. The equation solved in Professor Bell's note is that for the square of the line element in the general non-euclidean time-space, which holds a central position in Einstein's theory of gravitation. The solution is not free from quadratures. It is shown that particular solutions of certain other equations in general dynamics may be similarly found. The note will appear in the *Philosophical Magazine*.

B. A. BERNSTEIN,
Secretary of the Section.

THE SUM OF THE FACE ANGLES OF CERTAIN POLYHEDRONS IN n-SPACE.

BY DR. HARRIS F. MACNEISH.

(Read before the American Mathematical Society December 30, 1919.)

A SIMPLE polyhedron in n-space is defined as a set of r-spaces ${}_{n-i}C_{r-i}$ of which intersect in each i-space; $r > i$; $r = 1, 2, \cdots, n-1$; $i = 0, 1, \cdots, n-2$.

For a simple polyhedron P^n in n-space we denote the sum of the angles of all the plane faces by S_{p^n}, the number of

vertices by p_0^n, the number of line edges by p_1^n, and the number of r-space faces by p_r^n; $r = 2, 3, \cdots, n-1$.

For a simple polyhedron in n-space we develop the following formula for the sum of the face angles:

$$S_{p^n} = \left[\frac{1}{3}n(n-1)p_0^n - \frac{4}{n-2}p_3^n\right]\pi.$$

For space of three dimensions for any simple polyhedron P^3 we have the Euler formula

$$S_{p^3} = (p_0^3 - 2)2\pi.$$

In space of four dimensions we consider any simple polyhedron P^4. In any 3-space face A_i^3; $S_{A_i^3} = (a^3{}_{0,i} - 2)2\pi$. Taking the sum for all of the 3-space faces of P^4, since in a simple polyhedron four 3-space faces pass through each vertex, and two 3-space faces through each plane,

$$2S_{p^4} = (4p_0^4 - 2p_3^4)2\pi, \quad \text{or} \quad S_{p^4} = (2p_0^4 - p_3^4)2\pi.$$

To obtain an induction proof we assume the generalization of this formula for space of r-dimensions, i.e.,

$$S_{p^r} = \left[\frac{1}{3}r(r-1)p_0^r - \frac{4}{r-2}p_3^r\right]\pi.$$

In $(r+1)$-space the sum of the plane face angles of each r-space face P_i^r of a simple polyhedron P^{r+1} is given by the above formula.

Taking the sum of the plane face angles for all the r-space faces of the polyhedron P^{r+1}, since (a) $r+1$ r-space faces pass through each vertex; (b) $r-2$ r-space faces pass through each 3-space and (c) $r-1$ r-spaces pass through each plane, in a simple polyhedron in $(r+1)$-space

$$S_{p^{r+1}} = \left[\frac{1}{3}r(r+1)p_0^{r+1} - \frac{4}{r-1}p_3^{r+1}\right]\pi,$$

which completes the induction proof.

ON BORDERED FREDHOLM DETERMINANTS.

BY PROFESSOR T. H. HILDEBRANDT.

In a paper on projective transformations in function space, L. L. Dines* sets up certain expressions which he calls bordered Fredholm determinants, and minors. These determinants have properties which are analogous to the properties of Fredholm determinants. It is the purpose of the present note to point out that these bordered Fredholm determinants can be regarded as the Fredholm determinants of a system or a mixed system of linear integral equations, and that as a consequence their properties can be deduced from the corresponding properties of Fredholm determinants.

The bordered determinants arise in the question of the inversion of the transformation

$$(1) \qquad \psi(x) = \frac{\alpha(x) + \beta(x)\varphi(x) + \int_a^b \gamma(x, y)\varphi(y)dy}{\delta + \int_a^b \epsilon(y)\varphi(y)dy},$$

where δ is a constant, α, β, ϵ, φ and ψ are functions in the interval $I: a \leqq x \leqq b$, which we assume to be continuous, and $\gamma(x, y)$ is continuous in the square $S: a \leqq x \leqq b$, $a \leqq y \leqq b$. It is also assumed that β does not vanish in the interval I; as a consequence the question of the inversion of the transformation (1) is reducible to the inversion of the special transformation in which β is taken equal to unity on I. The existence of a unique inverse function φ for every ψ is equivalent to the existence of a unique solution of the system of equations

$$(2) \qquad \begin{aligned} \psi_1(x) &= \varphi_1(x) + \alpha(x)\varphi_2 + \int \gamma(x, y)\varphi_1(y)dy, \dagger \\ \psi_2 &= \delta\varphi_2 + \int \epsilon(y)\varphi_1(y)dy, \end{aligned}$$

where φ_2 and ψ_2 are constants and $\varphi = \varphi_1/\varphi_2$ and $\psi = \psi_1/\psi_2$. The determination of φ_1 and φ_2 from this system is in turn

* Cf. *Transactions Amer. Math. Society*, vol. 20 (1919), pp. 45–65.

† Here, as in the sequel, we omit the limits of integration (a, b).

equivalent to the question of solving the system of linear integral equations

$$\psi_1 = \varphi_1 + \int \kappa_{11}\varphi_1 + \int \kappa_{12}\varphi_2,$$

$$\psi_2 = \varphi_2 + \int \kappa_{21}\varphi_1 + \int \kappa_{22}\varphi_2,$$

where

$$\kappa_{11}(x, y) = \gamma(x, y), \qquad \kappa_{12}(x, y) = \frac{\alpha(x)}{b - a}, \qquad \kappa_{21}(x, y) = \epsilon(y)$$

and

$$\kappa_{22}(x, y) = \frac{\delta - 1}{b - a}.$$

On account of the nature of the functions κ_{12}, κ_{21}, and κ_{22}, every determinant of the Fredholm expansion of the system $\begin{pmatrix} \kappa_{11}\kappa_{12} \\ \kappa_{21}\kappa_{22} \end{pmatrix}$ which contains more than one row or column involving κ_{12}, κ_{21}, κ_{22} vanishes identically, and the expansion reduces to

$$1 + \Sigma_n \frac{1}{n!} \int \cdots \int |\gamma(x_i, x_j)| dx_1 \cdots dx_n + \Sigma_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \delta - 1 & \alpha(x_i) \\ \epsilon(x_i) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \qquad (i, j = 1, \cdots, n),$$

which on combination becomes

$$1 + \Sigma_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \delta & \alpha(x_i) \\ \epsilon(x_j) & \gamma(x_i, x_i) \end{vmatrix} dx_1 \cdots dx_n \quad (i, j = 1, \cdots, n),$$

which is exactly the expression for Dines's bordered determinant B.

B being then in fact a Fredholm determinant, it follows that we can derive the properties of B from those of Fredholm determinants. For instance, it is at once apparent that there is a product theorem between determinants B based on different functions. In particular since the product of the Fredholm determinants of the two systems $\begin{pmatrix} \kappa_{11}^{(1)}\kappa_{12}^{(1)} \\ \kappa_{21}^{(1)}\kappa_{22}^{(1)} \end{pmatrix}$ and $\begin{pmatrix} \kappa_{11}^{(2)}\kappa_{12}^{(2)} \\ \kappa_{21}^{(2)}\kappa_{22}^{(2)} \end{pmatrix}$ is the system

$$\begin{pmatrix} \kappa_{11}^{(1)} + \kappa_{11}^{(2)} + \int \kappa_{11}^{(1)}\kappa_{11}^{(2)} + \int \kappa_{12}^{(1)}\kappa_{21}^{(2)}, & \kappa_{12}^{(1)} + \kappa_{12}^{(2)} + \int \kappa_{11}^{(1)}\kappa_{12}^{(2)} + \int \kappa_{12}^{(1)}\kappa_{22}^{(2)} \\ \kappa_{21}^{(1)} + \kappa_{21}^{(2)} + \int \kappa_{21}^{(1)}\kappa_{11}^{(2)} + \int \kappa_{22}^{(1)}\kappa_{21}^{(2)}, & \kappa_{22}^{(1)} + \kappa_{22}^{(2)} + \int \kappa_{21}^{(1)}\kappa_{12}^{(2)} + \int \kappa_{22}^{(1)}\kappa_{22}^{(2)} \end{pmatrix}$$

we can substitute in this the corresponding values for the κ and find that the B-determinant of the functions δ_1, α_1, ϵ_1, γ_1 multiplied by the B-determinant of δ_2, α_2, ϵ_2, γ_2 is the B-determinant of $\delta_1\delta_2 + \int \epsilon_1\alpha_2, \alpha_2 + \int \gamma_1\alpha_2 + \alpha_1\delta_2, \epsilon_1 + \int \epsilon_1\gamma_2 + \delta_1\epsilon_2, \gamma_1 + \gamma_2 + \int \gamma_1\gamma_2 + \alpha_1\epsilon_2$, which is the result obtained by Dines by more cumbersome methods.

In order to complete the inversion problem it is necessary to introduce minors of the first order. There will be four of these in a system of two integral equations. If we set up these determinants for the special kernels under consideration and note as before that every determinant in the expansion which contains more than one row or column of the α, ϵ and δ is identically zero, we easily obtain the following expressions:

$$F_{11} = -\begin{vmatrix} \delta & \alpha(x) \\ \epsilon(y) & \gamma(x, y) \end{vmatrix} - \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \delta & \alpha(x) & \alpha(x_i) \\ \epsilon(y) & \gamma(x, y) & \gamma(x_i, y) \\ \epsilon(x_j) & \gamma(x, x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \qquad (i, j = 1, \cdots n),$$

$$F_{12} = -\frac{1}{b-a}\left[\alpha(x) + \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \alpha(x) & \alpha(x_i) \\ \gamma(x, x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n\right] \qquad (i, j = 1, \cdots, n),$$

$$F_{21} = -\epsilon(y) - \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \epsilon(y) & \gamma(x_i, y) \\ \epsilon(x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \qquad (i, j = 1, \cdots, n),$$

$$F_{22} = -\frac{1}{b-a}\left[\delta - 1 + \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \delta - 1 & \alpha(x_i) \\ \epsilon(x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n\right] \qquad (i, j = 1, \cdots, n).$$

By comparing with the results of Dines we see that

$$F_{11} = B_1, \qquad F_{12} = A/(b-a), \qquad F_{21} = E,$$

$$F_{22} = (D-B)/(b-a).$$

If now we write down the eight reciprocal relations governing the Fredholm determinant and its first minors for a system of equations, as for instance

$$F_0\kappa_{11} + F_{11} + \int F_{11}\kappa_{11} + \int F_{12}\kappa_{21} = 0$$

and seven others of a similar nature, and substitute therein the values for our special case, we get at once the reciprocal relations relative to B_1, A, E, B and D which Dines obtains. It is thus possible to make the discussion of the inversion of the projective transformation on the basis of the Fredholm determinants of a system of linear integral equations.

A more elegant approach to the same results is possible if we note that the system of equations (2) is a mixed system to which the results obtained by Moore* on general linear integral equations apply. Moore discusses not only the case when the functions which enter the equations have the same ranges but also when the ranges are different. Thus in the system under consideration, we really have two distinct ranges, the interval I: $a \leqq x \leqq b$, with the continuous functions φ_1, and ψ_1, and the operation of integration; and the range consisting of a single element with the functions φ_2, ψ_2, i.e., a single value, and the operation, the identity operation. By applying the methods sketched by Moore and reducing the results, we obtain at once as the Fredholm determinant of this system, the bordered determinant B, and as first minors, the expressions for B_1, A, E, and $D - B$. By substituting in the proper reciprocal relations, we arrive at the reciprocal relations obtained by Dines. This method has the advantage of not introducing any extraneous material, i.e., yielding at once the particular quantities which we desire.

This method is also useful in indicating a situation in which one obtains as Fredholm determinants a determinant bordered by n rows and columns. For if we replace the second range which consists of a single element, by a range which consists

* This BULLETIN, vol. 18 (1912), pp. 356 ff.

of a finite number m of elements, we obtain the equations

$$\psi(x) = \varphi(x) + \int \kappa(x, y)\varphi(y)dy + \Sigma_1^m \kappa_h(x)\varphi_h, \quad (3)$$

$$\psi_g = \varphi_g + \int \lambda_g(y)\varphi(y)dy + \Sigma_1^m k_{gh}\varphi_h,$$

where φ and ψ are defined on $a \leqq x \leqq b$, and φ_g and ψ_g on $(1, \cdots, m)$. When we set up the Fredholm determinant of this system, we find that all the determinants involving more than m rows or columns of the kernels κ_h, λ_g and k_{gh} vanish identically, and by collecting the non-vanishing elements properly we obtain without much trouble as the Fredholm determinant of this system the expansion

$$|\delta_{gh} + k_{gh}| + \Sigma_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \delta_{gh} + k_{gh} & \kappa_g(x_i) \\ \lambda_h(x_j) & \kappa(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \quad \begin{pmatrix} g, h = 1, \cdots, m \\ i, j = 1, \cdots, n \end{pmatrix},$$

where $\delta_{gh} = 0$ if $g \neq h$ and unity if $g = h$. Obviously the first and higher ordered minors and the relation between these can be written out without much difficulty.

Systems of linear equations of this last type arise in a number of connections. For instance the mixed linear integral equat on

$$\psi(x) = \varphi(x) + \int \kappa(x, y)\varphi(y)dy + \Sigma_1^m \kappa_g(x)\varphi(x_g),$$

where x_h, $h=1, \cdots, m$, are m points on the interval $a \leqq x \leqq b$, can be reduced to such a system. We need only adjoin the m equations which we obtain by putting $x = x_h$ in this equation and set $\varphi(x_g) = \varphi_g$ in order to obtain such a system. It would follow then that the Fredholm determinant of this mixed integral equation can be written in the form

$$|\delta_{gh} + \kappa_g(x_h)| + \Sigma_n \frac{1}{n!} \int \cdots \int \begin{vmatrix} \delta_{gh} + \kappa_g(x_h) & \kappa_g(x_i) \\ \kappa(x_h, x_j) & \kappa(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \quad \begin{pmatrix} g, h = 1, \cdots, m \\ i, j = 1, \cdots, n \end{pmatrix}$$

with $\delta_{gh} = 0$ if $g \neq h$ and $\delta_{gg} = 1$.

Another situation in which we get such a bordered Fred-

holm determinant is in the system of linear integral equations

$$\psi_1 = \varphi_1 + \int \kappa_{11}\varphi_1 + \int \kappa_{12}\varphi_2, \tag{4}$$
$$\psi_2 = \varphi_2 + \int \kappa_{21}\varphi_1 + \int \kappa_{22}\varphi_2,$$

in which the kernels κ_{12}, and κ_{22} are expressed in terms of the $3n$ functions $\alpha_i(x)$, $\beta_i(x)$, and $\gamma_i(x)$ as follows:

$$\kappa_{12} = \Sigma_1^n \alpha_i(x)\beta_i(y), \qquad \kappa_{22} = \Sigma_1^n \gamma_i(x)\beta_i(y).$$

If we substitute these values and multiply the second of the two equations by $\beta_j(x)$ and integrate with respect to x, we replace the above system by a system of the type (3) in which $\varphi_j = \int \beta_j(x)\varphi_2(x)dx$. The value of the function φ_2 can then be determined from the second of the equations (4). Since the Fredholm determinant of a system (3) is a bordered determinant, the Fredholm determinant of a system of the type of (4) would also be.

Obviously, these results can be extended by introducing the general range in place of the range I: $a \leqq x \leqq b$ and general classes of functions instead of continuous functions, and a general linear operator in place of integration in accordance with the postulates of Moore's general theory.

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ON THE COHERENCE OF CERTAIN SYSTEMS IN GENERAL ANALYSIS.

BY PROFESSOR A. D. PITCHER

IN a previous paper* attention was called to the property *coherence*, so named, of systems $(\mathfrak{Q}; \delta)$, where $\mathfrak{Q}$ is an abstract class of elements q and $\delta(q_1q_2)$ a distance function defined for every pair q_1q_2 of elements of $\mathfrak{Q}$. In the present paper it is shown that certain of the most notable of the systems of E. H. Moore's Introduction to General Analysis† which were devised without reference to coherence, or indeed without

* "On the foundations of the calcul fonctionnel of Fréchet," by A. D. Pitcher and E. W. Chittenden, *Transactions Amer. Math. Society*, vol. 19, No. 1, pp. 66–78.

† Cf. New Haven Mathematical Colloquium, Yale University Press, 1910. We refer to this memoir as I. G. A.

reference to any property of the range of the independent variable, must nevertheless be coherent.

Consider a system $(\mathfrak{P}; \Delta)$, i.e., an abstract class $\mathfrak{P}$ of elements p with a development Δ^*, and the relation $K_{p_1p_2m}$† to which the development Δ gives rise. Two sequences $\{p_{1n}\}$ $\{p_{2n}\}$ are said to be connected in case for every integer m there is an integer n_m such that $n \geqq n_m$ implies $K_{p_{1n}p_{2n}m}$. An element p is said to be a limit of a sequence $\{p_n\}$, in notation $L_n\, p_n = p$, in case for every m there is an n_m such that $n \geqq n_m$ implies $K_{p_np\,m}$.‡ A system $(\mathfrak{P}; \Delta)$, or a system $(\mathfrak{P}; K)$, is coherent if it is true that whenever two sequences $\{p_{1n}\}$ $\{p_{2n}\}$ are connected and p is a limit of the sequence $\{p_{1n}\}$ then p is also a limit of the sequence $\{p_{2n}\}$.

We wish to consider the systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$§ where the class $\mathfrak{M}$ has the properties $D_1\Delta K_2$ or the properties $L\Delta K_2$.‖

THEOREM. *The system* $(\mathfrak{P}; \Delta)$ *of a system* $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$, *where the class* $\mathfrak{M}$ *has the properties* $D_1\Delta K_2$ *or the properties* $L\Delta K_2$, *is a coherent system.*¶

Since $\{p_{1n}\}$ and $\{p_{2n}\}$ are connected there is a sequence $\{\mathfrak{P}^{\hat{m}_n\hat{l}_n}\}$ of the development Δ such that $L_n\hat{m}_n = \infty$ and such that for $n \geqq n_0$ the elements p_{1n} and p_{2n} each belong to the class $\mathfrak{P}^{\hat{m}_n\hat{l}_n}$. Since p is a limit of the sequence $\{p_{1n}\}$ there is a sequence $\{\mathfrak{P}^{\overline{m}_n\overline{l}_n}\}$, of the development Δ, such that $L_n\overline{m}_n = \infty$ and such that for n sufficiently large p and p_{1n} each belong to $\mathfrak{P}^{\overline{m}_n\overline{l}_n}$.

We must show that p is a limit of the sequence $\{p_{2n}\}$. Suppose this is not true. Then there is an integer m' such that for every n_0 there is an $n > n_0$ such that p and p_{2n} are not in the relation $K_{pp_{2n}m'}$. Thus by permitting n_0 to assume the values 1, 2, 3, . . . , k, . . . and choosing each time the smallest n available we secure a sequence $\{n_k\}$ such that for no k is the relation $K_{pp_{2n_k}m'}$ holding. There is then a sequence $\{p_{2n_k}\}$, a subsequence of $\{p_{2n}\}$, such that for no m exceeding

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* Cf. I. G. A., § 75.

† I. G. A., § 77.

‡ Cf. "A contribution to the foundations of the calcul fonctionnel of Fréchet," by T. H. Hildebrandt, *Amer. Jour. of Mathematics*, vol. 34 (1912), pp. 237–290.

§ I. G. A., §§ 75–80.

‖ For properties D_1, L, Δ, K_2 of classes $\mathfrak{M}$ see I. G. A., §§ 15, 22, 79, 72.

¶ It should be noted that the property D implies the property D_1 and on that account the systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$, where $\mathfrak{M}$ has the properties $D\Delta K_2$, are included in the above theorem.

m' and for no k are p_{2n_k} and p in the same subclass $\mathfrak{P}^{ml}$ of the development Δ.

Now consider the element p and those functions of the developmental system $((\delta^{ml}))$ which are associated with the element p. That is, consider the system $((\delta^{mq}))$ where $(\mathfrak{P}^{mq})$ is the system of classes of stage m which contain p. Denote by θ^m the sum of the functions δ^{mq} which are associated with p at stage m. Then, by the first condition* on the developmental system,

$$L_m\theta_p{}^m = 1.$$

In case $\mathfrak{M}$ has either the property L or the property D_1 the function θ^m has the property $K_2\mathfrak{M}$†. Consider a function φ which has the property $K_2\mathfrak{M}$. There is a μ of $\mathfrak{M}$ such that for every positive number e there is an m_e such that $K_{p_1p_2m_e}$ implies $|\varphi_{p_1} - \varphi_{p_2}| \leqq e\,|\mu_{p_1}|$. Since $\{p_{1n}\}$ and $\{p_{2n}\}$ are connected it follows that for every e there is an n_e such that $n \geqq n_e$ implies $|\varphi_{p_{1n}} - \varphi_{p_{2n}}| \leqq e\,|\mu_{p_{1n}}|$. Since the sequence $\{p_{1n}\}$ has p for a limit and μ has the property $K_2\mathfrak{M}$, the sequence $\{\mu_{p_{1n}}\}$ of values is bounded. Thus for every e there is an n_e such that $|\varphi_{p_{1n}} - \varphi_{p_{2n}}| \leqq e$. Therefore $L_n\varphi_{p_{1n}} = L_n\varphi_{p_{2n}} = \varphi_p$.

If the conclusion concerning φ be applied to each function θ^m, we have

$$L_n\theta_{p_{1n}}{}^m = L_n\theta_{p_{2n}}{}^m = \theta_p{}^m.$$

Then for every m

$$L_k\theta_{p_{2n_k}}{}^m = \theta_p{}^m$$

and

$$L_m(L_k\theta_{p_{2n_k}}{}^m) = L_m\theta_p{}^m = 1.$$

According to the condition (1b) on the developmental system there is for every μ a function μ_0 such that for every e there is an m_e such that for $m \geqq m_e$ and for every p

$$\Sigma_h\,|\,\mu_{r^{mh}}\delta_p{}^{mh}\,| \leqq e\,|\,\mu_{0p}\,|.$$

Consider an element p_{2n_k}. Recall that $\theta^m = \Sigma_q\delta^{mq}$. For m sufficiently large no δ^{mq} is associated with a class $\mathfrak{P}^{ml}$ to which p_{2n_k} belongs (since δ^{mq} is associated with a class $\mathfrak{P}^{mq}$ to which p belongs). Since $\mathfrak{M}$ has the properties L, Δ, K_2 or the properties D_1, Δ, K_2 there is a function μ of $\mathfrak{M}$ which, for m sufficiently large and a number a properly chosen, is such

* I. G. A., § 78 (1a).
† I. G. A., § 72 (2).

that $a\mu_{r^{mq}} \geqq 1$, where r^{mq} is the representative element of any class $\mathfrak{P}^{mq}$ to which p belongs. Using such a function μ in the condition (1b) on the developmental system and noting that the sequence $\{\mu_{0p_{2n_k}}\}$ is bounded, we see that for every e there is an m_e such that for $m \geqq m_e$ and for every k the value $\theta_{p_{2n_k}}{}^m$ does not exceed e. But this affords a contradiction to the conclusion reached above that

$$L_m(L_k\theta_{p_{2n_k}}{}^m) = 1.$$

Therefore the hypothesis that p is not a limit of the sequence $\{p_{2n}\}$ is contrary to fact.

These considerations may be extended to the infinite developments of Chittenden.*

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ON THE ADJOINT OF A CERTAIN MIXED EQUATION.

BY DR. R. F. BORDEN.

(Read before the American Mathematical Society December 30, 1919.)

CONSIDER a function of the form

$$(1)\quad F\{f(x)\} = \Delta f'(x) + a(x)f'(x) + b(x)\Delta f(x) + c(x)f(x),$$

where $a(x)$, $b(x)$, and $c(x)$ are analytic functions of x, also $\Delta f(x) = f(x+1) - f(x)$, and $f'(x) = (d/dx)f(x)$. We will say that $G\{g(x)\}$ is the adjoint of $F\{f(x)\}$ if $G\{g(x)\} = 0$ is the condition that

$$(2)\quad \int \Sigma\, g(x)F\{f(x)\}dx = \Sigma M(x) + \int N(x)dx,$$

where Σ denotes an inverse of Δ.

This condition (2) is satisfied if

$$(3)\quad g(x)F\{f(x)\} = \frac{d}{dx}M(x)dx + \Delta N(x),$$

* Cf. "Infinite developments and the composition property $(K_{12}B_1)_*$ in general analysis," by E. W. Chittenden, *Rendiconti del Circolo Matem. di Palermo*, vol. 39 (1915), p. 21, §§ 19 and 21.

where $M(x)$ and $N(x)$ are of the form

$$(4)\qquad \begin{aligned} M(x) &= A(x)\Delta f(x) + B(x)f(x), \\ N(x) &= C(x)f'(x) + D(x)f(x). \end{aligned}$$

Equating corresponding coefficients in the two members of (3), we have

$$(5)\qquad \begin{aligned} g(x) &= A(x) + C(x), \\ a(x)g(x) &= B'(x), \\ b(x)g(x) &= A'(x) + D(x), \\ c(x)g(x) &= D'(x), \\ \Delta C(x) &= \Delta D(x) = 0. \end{aligned}$$

From (5) we have

$$(6)\qquad \begin{aligned} &\Delta g'(x) - a(x)g'(x) - b(x+1)\Delta g(x) \\ &\qquad + [c(x) - a'(x) - \Delta b(x)]g(x) = 0, \end{aligned}$$

which we will call the expression adjoint to the expression (1).* The equations formed by equating (1) and (6) to zero, we will call adjoint equations.

The formal mode of obtaining the coefficients of the adjoint gives (1) as the adjoint to (6) only when $\Delta b(x) = 0$. However this difficulty may be avoided, for $b(x)$ may be taken identically zero without loss of generality, since (1) is transformed by the substitution

$$f(x) = e^{-\int_{x_0}^{x} b(x)dx} h(x)$$

into an expression having no term in $\Delta h(x)$. Accordingly in what follows we will take $b(x)$ as zero when discussing the adjoint.

We will now state a few facts about the equation†

$$(7)\qquad \Delta f'(x) + a(x)f'(x) + b(x)\Delta f(x) + c(x)f(x) = 0,$$

where $a(x)$ is not identically one.

* Other forms of adjoint expressions may be found to satisfy (3), but among other advantages (6) affords in what follows the closest analogy to the theory of adjoint partial differential equations. See, for instance, Forsyth: Theory of Differential Equations, Vol. 6, p. 112.

† First studied by Poisson: *Journal de l'Ecole Polytechnique*, vol. 6. (1806), pp. 127–141. Its properties here listed are developed in my thesis which is to appear in the *Amer. Jour. of Mathematics*.

(i) The equation (7) has under the group of transformations $f(x) = v(x)g(x)$ two fundamental invariants

$$I(x) = \frac{c(x) - [a(x) - 1]b(x) - a'(x)}{a(x) - 1}$$

and

$$J(x) = \frac{c(x) - [a(x) - 1]b(x - 1)}{a(x) - 1}.$$

(ii) When either $I(x) = 0$ or $J(x) = 0$, the equation may be solved in finite form by the standard methods of solving differential and difference equations, the two processes being separable.

(iii) If n successive applications of the transformation

$$(S) \qquad f_{S_1}(x) = f(x + 1) + [a(x) - 1]f(x)$$

or of the transformation

$$(T) \qquad f_{T_1}(x) = f'(x) + b(x - 1)f(x)$$

result in an equation with a vanishing invariant, and if n is the least integer for which this is true, then the equation is of rank $n + 1$ with respect to the transformation.

(iv) If I_{S_m} and J_{S_m} are the invariants of the mth transformed equation under (S), and if I_{T_m} and J_{T_m} are the invariants of the equation resulting from m applications of I, then

$$J^S_n(x) = I_{S_{n-1}}(x) \text{ and } I_{T_n}(x) = J_{T_{n-1}}(x).$$

(v) The equation obtained by applying (S) and (T) in the order named, has the same invariants as has the equation obtained by applying first (T) and then (S).

The adjoint to

$$(8) \qquad \Delta f'(x) + a(x)f'(x) + c(x)f(x) = 0$$

is

$$(9) \qquad \Delta g'(x) - a(x)g'(x) + [c(x) - a'(x)]g(x) = 0,$$

which has the invariants*

$$i(x) = \frac{c(x) - a'(x)}{-a(x) + 1} + \frac{a'(x)}{-a(x) + 1} = \frac{c(x)}{-a(x) + 1} = -J(x)$$

* The discussion holds only when $a(x) \neq 1$.

and

$$j(x) = \frac{c(x) - a'(x)}{-a(x) + 1} = -I(x).$$

Hence the adjoint equation has the same set of invariants except for order as has the original equation. Consequently *if one of two adjoint equations of the form* (8) *can be solved in finite form on account of a vanishing invariant, the same is true of the other.*

Denote equation (8) by $F = 0$ and its nth transform by $F_{S_n} = 0$ or by $F_{T_n} = 0$ according as (S) or (T) is used. Using (iv) and (v) we can express the invariants of the transformed equations as follows, each pair of invariants being written under the corresponding equation:

$$\begin{array}{llllll} F_{T_n} = 0, & \cdots, & F_{T_1} = 0, & F = 0, & F_{S_1} = 0, & \cdots, F_{S_n} = 0, \\ I_{T_n}, & \cdots, & I_{T_1}, & I, & I_{S_1}, & \cdots, I_{S_n}, \\ I_{T_{n+1}}, & \cdots, & I_{T_2}, & I_{T_1}, & I, & \cdots, I_{S_{n-1}}. \end{array}$$

Similarly we have for the adjoint equation $G = 0$, the set

$$\begin{array}{llllll} G_{T_n} = 0, & \cdots, & G_{T_1} = 0, & G = 0, & G_{S_1} = 0, & \cdots, G_{S_n} = 0, \\ I_{S_{n-1}}, & \cdots, & I, & I_{T_1}, & I_{T_2}, & \cdots, I_{T_{n+1}}, \\ I_{S_n}, & \cdots, & I_{S_1}, & I, & I_{T_1}, & \cdots, I_{T_n}, \end{array}$$

which differs from the set for $F = 0$ and its transforms only in order.

From the two sets we see that, *if an equation of the form* (8) *is of finite rank with respect to either* (S) *or* (T), *its adjoint is of the same rank with respect to the other transformation. Also if such an equation is of doubly finite rank, its adjoint is of doubly finite rank with the same index.*

The equation

$$\Delta f'(x) + c(x)f(x) = 0 \tag{10}$$

is self-adjoint. Its invariants are equal and have the value $-c(x)$. Unlike the corresponding partial differential equation, (10) is not a canonical form for equations with equal invariants. A necessary and sufficient set of conditions that (7) shall have equal invariants is

$$a'(x) = \Delta b(x) = 0.$$

Not all equations of that form are reducible to the form (10); for example

$$\Delta f'(x) + 2f'(x) + c(x)f(x) = 0$$

is not. However, equations with equal invariants which are not self-adjoint do not seem to be of great interest.

From the preceding theorem it follows that *if a self-adjoint equation of form* (10) *is of finite rank with respect to one of the transformations* (S) *or* (T), *it is of the same rank with respect to the other.*

Using the formula*

$$I_{S_n}(x) = I(x) + \sum_1^{n-1}[I(x+k) - J(x+k)] - \Delta\frac{d}{dx}\log\left[\prod_{k=0}^{n-1} I(x+k)\prod_{k=0}^{n-2} I_{S_1}(x+k)\cdots I_{S_{n-1}}(x)\right]$$

and noting that for the self-adjoint case

$$I(x) = J(x) = -c(x),$$

we have

$$-c(x) = \Delta\frac{d}{dx}\log\left[\prod_{k=0}^{n-1}(-c(x+k))\prod_{k=0}^{n-2} I_{S_1}(x+k)\cdots I_{S_{n-1}}(x)\right]$$

as a necessary and sufficient condition that the self-adjoint equation (10) *be of rank* $n+1$ *with respect to each of the transformations* (S) *and* (T).

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THE SECOND VOLUME OF VEBLEN AND YOUNG'S PROJECTIVE GEOMETRY.

Projective Geometry. By OSWALD VEBLEN and J. W. YOUNG. Boston, Ginn and Company; Vol. 2, by Oswald Veblen, 1918. 12 + 511 pages.

IN volume I, Veblen and Young were concerned particularly with those theorems of projective geometry which can be proved on the basis of their assumptions A of alignment, assumptions E of extension, and an assumption P of pro-

* Formula (8) in my thesis, l.c.

jectivity. In some cases use was also made of the assumption H_0 that the diagonal points of a complete quadrangle are non-collinear. A space satisfying A and E is called a general projective space. This term does not seem to the reviewer to be altogether appropriate. According to this terminology real projective spaces, complex projective spaces, etc., are all general projective spaces. This seems rather like saying that every special projective space is a general projective space. It would seem in some respects preferable to omit the word general and say simply that every space satisfying A and E is a projective space.*

In volume II various special projective spaces are studied, each special projective space being a space in which, in addition to A and E, certain other postulates are satisfied. One example of a special projective space is a proper projective space, a space satisfying A, E and P. A projective space satisfying the assumption H that if any harmonic sequence exists not every one contains only a finite number of points is called a non-modular projective space. A modular projective space is one in which every harmonic sequence contains only a finite number of points.

Assumptions A, E, P and H_0 hold true in both the ordinary real and the ordinary complex projective spaces of three dimensions. A space satisfying these assumptions is not necessarily continuous in the sense of Dedekind. Indeed it does not necessarily possess even pseudo-archimedean continuity. Veblen and Young's assumptions A, E and H_0 correspond rather closely to the first fourteen of a set of nineteen postulates published by Pieri in 1899.† The following is a rough translation of Pieri's set.

POSTULATE I°. *Projective points form a class.*

POSTULATE II°. *There exists at least one projective point.*

POSTULATE III°. *If there exists one projective point, there exists at least one other one.*

POSTULATES IV° AND V°.‡ *If a, b are projective points ($a \neq b$), the line ab is a class of points.*

* The same objections would not apply to the use of the term general projective geometry to designate the totality of all theorems that can be proved on the basis of A and E. There is only one such body of theorems and therefore only one such geometry, while there are many sorts of spaces satisfying A and E.

† M. Pieri, "I principii della geometria di posizione composti in sistema logico deduttivo," *Memorie della Reale Accademia delle Scienze di Torino*, vol. 48 (1899), pp. 1–62.

‡ Cf. A_1.

POSTULATE VI°. *If a and b are distinct projective points, the line ab is contained in the line ba.*

POSTULATE VII°. *If a and b are distinct projective points, a belongs to the line ab.*

POSTULATE VIII°.* *If a and b are distinct projective points, the line ab contains at least one point distinct from a and from b.*

POSTULATE IX°. *If a and b are distinct projective points and c is a point of the line ab distinct from a, then b belongs to the line ac.*

POSTULATE X°. *Under the same hypothesis ab contains ac.*

POSTULATE XI°.† *If $a \neq b$, there exists at least one point not on the line ab.*

POSTULATE XII°.‡ *If a, b, c are non-collinear projective points and a′ is a point of bc distinct from b and from c and b′ is a point of ac distinct from a and from c, then the line aa′ has a point in common with the line bb′.*

POSTULATE XIII°.§ *If a, b, c are non-collinear points, there exists a point not in the plane abc.*

POSTULATE XIV°.‖ *If $a \neq b$ and c is a point of ab distinct from a and from b, then the fourth harmonic of c with respect to a and b is distinct from c.*

POSTULATE XV°. *If a, b, c are distinct points of a line r and d is a point of the line r not belonging to the segment¶ (abc) nor coinciding with a or with c, then d belongs to the segment (bca).*

POSTULATE XVI°. *If a, b, c are distinct collinear points of a line r and a point d belongs to both of the segments (bca) and (cab), then it does not belong to the segment (abc).*

POSTULATE XVII°. *If a, b, c are distinct collinear points and d is a point distinct from b on the segment (abc) and e is a point of the segment (adc), then the point e is on the segment (abc).*

Postulate XVIII° is a form of Dedekind cut postulate as applied to the points of a segment.

POSTULATE XIX°.** *If a, b, c, d are distinct non-coplanar*

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* Cf. Veblen and Young's E_0.

† Cf. Veblen and Young's E_2.

‡ Cf. Veblen and Young's A_3.

§ Cf. E_3.

‖ Cf. Assumption H_0. Pieri points out that in the presence of Postulates VI°, XII° and XIII°, Postulate XIV° is equivalent to the proposition that the diagonal points of a complete quadrangle are collinear.

¶ The *segment* (*abc*) is defined as the set of all points [*x*] such that, for some pair of points *y* and *z* on the line *abc* which are harmonic conjugates of each other with respect to *a* and *c*, *x* is the fourth harmonic of *b* with respect to *y* and *z*.

** Cf. Assumption E_3'.

points, then for every point e not in any one of the planes abc, abd, acd, bcd there exists at least one point common to the figures ae, bcd.

It seems not far from accurate to say that Veblen and Young's set A, E, H_0 differs from Pieri's set I°–XIV°, XIX° chiefly in that Veblen and Young postulate simply the four assumptions A_1, A_2, E_0 and E_1 instead of Pieri's I–X, which latter may be (perhaps rather roughly) considered as corresponding to the result of breaking up (or analyzing) the group of assumptions A_1, A_2, E_0 and E_1, in a certain fashion, into a larger number of weaker postulates. Every space that satisfies A, E, H_0 satisfies also I°–XIV° and conversely.

Veblen secures a categorical set of postulates for real projective space by defining the notion of order on a net of rationality and the notion of open cuts in such a net and adding, to A and E, assumption H and the following assumptions C and R.

ASSUMPTION C. *If every net of rationality contains an infinity of points, then on one line l in one net R* ($H_0H_1H_\infty$) *there is associated with every open cut* (A, B) *with respect to the scale* H_0, H_1, H_∞, *a point* $P_{(A,B)}$ *which is on l and such that the following conditions are satisfied*:

(1) *If two open cuts* (A, B) *and* (C, D) *are distinct, the points* $P_{(A,B)}$ *and* $P_{(C,D)}$ *are distinct*;

(2) *If* (A_1, A_2) *and* (B_1, B_2) *are any two cuts and* (C_1, C_2) *any open cut between two points A and B of R* ($H_0H_1H_\infty$) *and if T is a projectivity such that*

$$T(H_\infty AB) = H_\infty P_{(A_1, A_2)} P_{(B_1, B_2)},$$

then $T(P_{(C_1, C_2)})$ *is a point associated with some cut* (D_1, D_2) *between* (A_1, A_2) *and* (B_1, B_2).

ASSUMPTION R. *On at least one line, if there is one there is not more than one chain.*

Pieri secures the same result by defining the notion of a segment and adding postulates XV°–XVIII°.

It seems to the reviewer that one might naturally feel, in view of the complicated nature of Veblen's assumption C, that Pieri's XV°–XVIII° constitute a simpler group of assumptions than that constituted by Veblen's H, C and R. Veblen's set has an apparent advantage in that his assumption C holds true for a complex space as well as for a real one, while this is not true of Pieri's postulate XV°. But let us consider

a second article by Pieri, published in 1905.* In this article Pieri gives a set of postulates for complex projective geometry of infinitely many dimensions. The first 10 of these correspond closely to the first 14 of his first set except for the fact that (*a*) instead of XI° and XIII° of set I we have† in set II a postulate to the effect that for every complex space S_n there exists at least one point not in S_n and (*b*) postulate XIV° is replaced by a postulate to the effect that the diagonal points of a complete quadrangle are non-collinear.‡

In postulate XII and in most of the remaining postulates XII–XXX use is made of a new undefined notion, the notion of a chain. Postulates XII–XIX are (freely translated) as follows:

POSTULATE XII. *If a, b, c are distinct points of a complex line, the chain of the points a, b, c—indicated by the symbol* $|abc|$*—is a class of points belonging to that line.*

POSTULATES XIII AND XIV. *If a, b, c are distinct collinear complex points, the chain* $|abc|$ *is contained in each of the chains* $|acb|$ *and* $|bac|$.

POSTULATE XV. *Under the same hypothesis a belongs to the chain* $|abc|$.

POSTULATE XVI. *If a, b, c, d are distinct collinear points and a′, b′, c′, d′ are distinct collinear points in perspective correspondence with a, b, c, d, then d belongs to* $|abc|$ *if and only if d′ belongs to* $|a'b'c'|$.

POSTULATE XVII. *If a, b, c are three distinct collinear points, the fourth harmonic of c with respect to a and b belongs to the chain* $|abc|$.

POSTULATES XVIII AND XIX. *If a, b, c, d are four distinct collinear points and d belongs to the chain* $|abc|$, *then c belongs to* $|abd|$ *and* $|abd|$ *is contained in* $|abc|$.

If in the definition of "segment (*abc*)" quoted above for real space the phrase "line *abc*" is replaced by "chain $|abc|$" the resulting definition holds good for complex space.

Postulate XX of set II is equivalent to the proposition obtained by substituting "chain $|abc|$" for the second "line *r*" in the statement of postulate XV° of set I. Postulates XXI and XXII of set II are respectively equivalent to XVI° and XVII° of set I.

* "Nuovi principii di geometria projettiva complessa," *Memorie della Reale Accademia delle Scienze di Torino*, vol. 55 (1905), pp. 189–235.

† A natural change to make in passing from a set of postulates for three dimensions to one for infinitely many dimensions.

‡ Cf. Veblen and Young's H_0.

The set of postulates I–XXII hold true in real as well as in complex space. If, in this set, postulate X is replaced by XI°, XIII° and XIX° of set I, and Veblen's R and a suitable Dedekind cut postulate are added, there results a categorical set of axioms for real projective geometry of three dimensions. To obtain complex geometry, Pieri adds, to I–XXII, seven more postulates (XXIII–XXX). Of these, XXIII is a sort of pseudo-archimedean axiom whose content may be roughly suggested by saying that it postulates that if a, b, c are three distinct points the net of rationality determined by them is everywhere dense on the segment (abc).

At first sight it might seem that Veblen and Young's set of assumptions for complex projective geometry contains fewer postulates and is correspondingly simpler than Pieri's set. However, leaving out of consideration some of the earlier axioms in which Pieri gives such a detailed analysis of certain simple relations of alignment, etc., the question arises whether if Veblen's set seems at first sight to be simpler it is not largely because he makes use of a postulate, assumption C, which leaves much to be desired from perhaps at least two points of view. There is a curious difference between the way in which assumption C functions for complex space as compared with the way it functions for real space. Suppose S is a definite real projective space (satisfying A, E and H) whose projective structure is determined in the sense that the question whether or not three points in S lie on the same line has a determinate answer as soon as the three points are themselves determined. If C and R are satisfied in S for one association of points with open cuts as indicated in the statement of assumption C then they are not satisfied for any other such association of points with cuts. That is to say if, in S, (a) on one line l in one net $R(H_0H_1H_\infty)$ there is associated with every open cut (A, B) in $R(H_0H_1H_\infty)$ a point P on l such that conditions (1) and (2) of assumption C are satisfied and such that R is also satisfied, and (b) in S, on the same line l in the same net $R(H_0H_1H_\infty)$ there is associated with every open cut (A, B) (with respect to the scale H_0, H_1, H_∞) a point $\overline{P}_{(A,B)}$ on l satisfying the same conditions (1), (2) and R, then, for every (A, B), $\overline{P}_{(A,B)}$ is identical with $P_{(A,B)}$.

This uniqueness of choice of the association in question does not however exist for the case of a complex space satisfying A, E, H, C, $\overline{R}$ and I. Hence it is not determined in advance

just what point must be associated with each cut. Indeed the reviewer believes that in this case for a given H_0, H_1, H_∞ on a given line l and a given open cut $(\bar{A}, \bar{B})$ on l not related in a certain way to H_0, H_1, and H_∞, one may select, for the $P_{(\bar{A}, \bar{B})}$ to be associated with $(\bar{A}, \bar{B})$ *any point whatsoever* of the line l with the exception of an infinity of points which are (in a certain sense) related algebraically to the points H_0, H_1, H_∞—provided of course that after this selection is made, the points to be assigned to the other open cuts are properly chosen. Thus there appears to be involved here an arbitrary element of a very pronounced character. If the author claims that in using assumption C for a complex space he does not introduce a new undefined idea, then the reviewer would like to ask the following questions.

Why do the authors of volume I say, on page 1 of that volume, "Since any defined element or relation must be defined in terms of other elements and relations, it is necessary that one or more of the elements *and** one or more of the relations between them remain entirely *undefined*†; otherwise a vicious circle is unavoidable" and why, on page 15 of volume 1 do they say "We consider a class (cf. § 2, page 2) the elements of which we call *points*, and certain undefined *classes of points* which we call lines"? Why not say merely, "We consider a class the elements of which we call points" and substitute the following in place of the treatment beginning with line 9 of page 16?

"Concerning points we now make the following assumption:

"ASSUMPTION A. With every two points X and Y there is associated a class of points XY such that

(1)‡ if A, B and C are points which do not belong to XY for any two points X and Y, and D and E $(D \neq E)$ are points such that B, C, D belong to X_1Y_1 for some pair of points X_1 and Y_1 and C, A, E belong to X_2Y_2 for some pair of points X_2Y_2 then there is a point F such that A, B, F belong to some X_3Y_3 and D, E, F belong to some X_4Y_4,

(2)§ If $X \neq Y$ there are at least three points in the class XY.

(3)‖ There do not exist two distinct points X and Y such that every point is in the class XY."

Etc.

Can any objection be made to this procedure which

* Italics are the reviewer's.

† Italics are the authors'.

‡ Cf. A_3. § Cf. E_0. ‖ Cf. E_3.

would not also be an objection to the employment of C as an assumption for complex geometry? If in using such an assumption as C the author does not introduce a new undefined relation, then is it not at least true that in almost (or quite) every case where a set of postulates has ever been constructed in terms of more than one undefined element one can construct a closely corresponding set in terms of only *one* undefined element in a manner similar to that employed in C and in the example that I have given above?

A rather puzzling question now arises. If it be granted that in employing C in the set $A, E, H, C, \bar{R}, I$, the author is really introducing a new undefined relation then is it or is it not true that this is also the case for the set A, E, H, C, R? Or does the mere fact that the above-mentioned arbitrary element is present in the first case and absent in the second—does this fact alone afford sufficient grounds for concluding that the use of C introduces a new undefined relation in the first connection but not in the second?

Aside from the above considerations, assumption C leaves something to be desired from the standpoint of analysis. And what are we to understand is the relation between C, $\bar{R}$ and I? Should $\bar{R}$ be stated as a *separate* assumption? It reads "On some line l not all points belong to the same chain." Here we have a postulate which is stated separately from C but in which there is used a term (chain) which has been defined (?) in terms of an "association" postulated in C. It seems to me the situation would have been made clearer if instead of stating $\bar{R}$ and I separately the author had incorporated in postulate C, after the statement of condition (2), something like the following:

"(3) If in terms of this association of points with cuts 'the chain defined by A, B, C' is defined as indicated on pages 17 and 21 for any three collinear points A, B, C; then there exists a line $\bar{l}$ such that (a) on $\bar{l}$ not all points belong to the same chain, (b) through a point P of any chain C of the line $\bar{l}$ and any point J on $\bar{l}$ but not in C there is not more than one chain on $\bar{l}$ which has no other point than P in common with C."

Of course this strengthened postulate C is more complicated than the original. But does not the attempt to state $\bar{R}$ and I as separate assumptions serve to becloud the true state of affairs?

In an article in the *Annals of Mathematics*, volume 11 (1909), page 34, J. W. Young says "The notion of a chain has been fundamental in the synthetic introduction of imaginaries into geometry since the time of von Staudt. In the more recent work on the foundations of projective geometry it necessarily plays an important rôle. Pieri has indeed recently chosen it as one of the undefined elements in his set of assumptions for complex geometry. More recently Professor Veblen and I have given a set of assumptions for projective geometry in which point and an undefined class of points called a line are the only undefined elements and in which the chain is defined." It is perhaps superfluous to say that the reviewer does not agree with this statement. First there at least seems to be room for debate on the question whether Veblen and Young do not employ a third undefined element. Secondly I certainly do not feel that they have defined chain anywhere in volume I or volume II. If it is true that point and line are the only undefined elements here and that chain is defined, then it would have been possible to have gotten along with point as the only undefined element, both line and chain being defined. One could* even do this on the basis of the single axiom that the cardinal number of the set of all points is C. In this case there would be an arbitrary element involved in the "definitions" of both line and chain. On what grounds if any can one object to this proceedure without objecting to that of Veblen and Young? If we are to allow an arbitrary element in connection with the notion of a chain on what grounds can we object to doing an entirely similar thing in connection with the notion of a line? Perhaps it may be contended that it is undesirable that this sort of arbitrariness should be associated with the notion of a line because this notion is so fundamental in projective geometry. In complex projective geometry the notion of a chain is also quite fundamental.† Indeed I am not sure but that it might be considered just as fundamental as that of a line. It seems to the reviewer that there is much to be said in favor of Pieri's treatment in which point, line and chain are undefined. The question arises however whether, from assumptions C, $\overline{R}$ and I and the results that the authors have established concerning them, one may not perhaps obtain *suggestions* for a well

* See below.

† Cf., indeed, the above quotation from one of the authors of Volume I.

analyzed set of postulates in terms of point, line and chain or in terms of point, line and order, but simpler than that of Pieri.

On pages 302, 303 and 304 occur three statements which for purposes of reference I will call I, II and III. These statements are as follows:

I. "Assumptions I–IX, XVII are categorical for the euclidean space, i.e., if two sets of objects $[P]$ and $[Q]$ satisfy the conditions laid down for points in the assumptions, there is a one-to-one reciprocal correspondence between $[P]$ and $[Q]$ such that the subsets called lines of $[P]$ correspond to the subsets called lines of $[Q]$. Thus the internal structure of a euclidean space is fully determined by assumptions I–IX, XVII."

II. "Assumptions I–X, XVII have a different rôle from X–XVI or XVIII–N in that they determine the set of objects (points, lines, etc.) which are presupposed by all the other assumptions. The choice of these assumptions is logically arbitrary. The choice of such sets of "assumptions" as X–XVI is not arbitrary, it must correspond to a properly chosen group of permutations of the objects determined by I–X, XVII."

III. "The point of view of the writer is that if X–XVI or XVIII–N are to be regarded as independent assumptions their independence is of a lower grade than that of I–IX, XVII. They constitute a definition by postulates of a relation (congruence or nearness) among objects (points, lines, etc.) already fully determined."

As to I, what is meant by the "internal structure" of a euclidean space? Do not circles have as much to do with the internal structure of euclidean space as do straight lines? If so it seems to me that the internal structure of a euclidean space is not at all fully determined by assumptions I–X, XVII. What does it mean to say that two sets of objects $[P]$ and $[Q]$ satisfy the conditions laid down for points in the assumptions? The assumptions in question are in terms of point and order (not point alone). In order to give a definite interpretation of a space satisfying I–IX, XVII do we not need to be told just what objects are points for that interpretation and just what points A, B and C are in the order ABC in that interpretation? Grant that we are given such an interpretation S of a space satisfying assumptions I–X, XVII. Then for any three points it is fully determined

whether or not they are on the same line* and to that extent the internal structure of the space S is fully determined. But if one who is curious to know whether the interpretation in question is an interpretation of euclidean geometry should pick out three non-collinear points and ask "are these three points on a circle?"—What answer could be made? One might reply (?) "if you pick out a polar system at infinity I will tell you whether A, B and C are on a circle with respect to that polar system"—but would that be an answer to the question?

As to II and III, if the mere fact that a certain set of postulates is categorical in terms of something, no matter what (I suppose the statement "they determine the set of objects," etc., means that I–X, XVII are categorical in terms of point and order), implies that no further postulates are necessary for any purpose or at least that any further postulates are independent to a lesser degree, if at all, then why not base all of euclidean geometry on the following axiom?

AXIOM A. *The number of points in space is C (the power of the continuum).*

On the basis of Axiom A, straight lines, order and congruence can be so defined that Axioms I–IX of § 29 and Axioms† X–XVI of § 66 will all be fulfilled. Thus Axiom A would be a sufficient basis for three-dimensional euclidean geometry. But straight lines, order and congruence could also be defined in some other way so that all the theorems of two-dimensional euclidean geometry would be fulfilled. With the use of another set of definitions Bolyai-Lobachevskian geometry could be obtained, etc., etc.

With reference to the last two sentences of Statement II, —in what sense, if any, are assumptions I–X, XVII arbitrary? Must not one select II so that it will not contradict I or follow from I, select III so that it will not contradict I or II or follow from them, etc.? Is anything different true of X–XVI? Are they not arbitrary except in that they must be selected so as not to contradict each other or I–X, XVII, and so that no one of them will follow from the others together with I–X, XVII? In a certain sense it may of course be said

* In § 29 the notion of a line is defined in terms of point and order.

† Cf. "Foundations of Geometry," by Oswald Veblen, in Monographs on Modern Mathematics, edited by J. W. A. Young, New York, 1911. Also R. L. Moore, "Sets of metrical hypotheses for geometry," *Trans. Amer. Math. Society*, vol. 9 (1908), pp. 487–512.

that every time a new (independent) assumption is added there is a decrease in the arbitrariness involved in the selection of an additional assumption (unless one is to have either dependence or a contradiction) and after one has postulated I–X, XVII there is of course less arbitrariness than there was at the beginning. It is true that no assumption can be added to I–X, XVII (without there being a contradiction) if it gives additional information (of a certain type*) concerning point and order alone. But it is allowable to add assumptions that give other information.

On page 71 Veblen narrows a definition of Klein by "assigning to the geometry corresponding to a given group only the theory of those properties which, while invariant under this group, are *not invariant under any other group of projective collineations containing it.*" He adds "This will render the question definite as to whether a given theorem belongs to a given geometry." Unless the reviewer fails to understand the meaning of these statements it appears that according to this test it is not a theorem of euclidean geometry that in a given plane there is only one line parallel to a given line through a point not on that line, nor is it a theorem of euclidean geometry that through two given points there is only one line, and it is not a proposition of double elliptic geometry that every two coplanar lines have two points in common. Apparently the author himself does not always hold to this point of view. For instance on page 70 he says "We have thus considered only very general properties of figures and so have dealt hardly at all with the familiar relations, such as perpendicularity, *parallelism*,† congruence of angles and segments, which make up the bulk of elementary euclidean geometry."

On page 83 in order to secure a complete definition of a planar field of vectors should not the author add to (1) and (2) the stipulations that (3) for each vector V and point A there is only one point B such that V corresponds to the ordered point pair AB and (4) if two ordered point pairs AB and $A'B'$ are not equivalent under the group of translations then their corresponding vectors are distinct? The definition as it stands seems to be satisfied if the number 1 is taken as the vector of *every* point pair.

The present review has been much concerned with a dis-

* Cf. E. H. Moore, "On the foundation of mathematics," *Science*, vol. 17 (1903), 401–416.

† Italics are the reviewer's.

cussion of certain more or less debatable and delicate questions relating to the foundations of mathematics. Let it be not imagined that these questions bear in any way on the usefulness and interest of the main body of the treatise under review.

In Chapter II the author considers order relations in real projective space. Chapter III is concerned with affine plane geometry, that is to say with the geometry corresponding to the group of all those projective collineations which transform into itself the set of all points not lying on a given line l_∞ of a given projective plane π. One of the most interesting features of the text under review is the way in which various propositions are classified under particular groups. For instance, though of course the affine group doesn ot leave all metrical properties invariant it does leave certain particular ones invariant and in Chapter III the author considers a considerable body of such properties. In particular it is interesting that a theory of equivalence of triangles can be based* on this group.

Chapter IV is concerned with euclidean plane geometry regarded as the geometry corresponding to the group of all those projective collineations that leave invariant a fixed involution (without double points) on a line l_∞ in a real projective plane. Chapter V is concerned with ordinal and metrical properties of conics; Chapter VI with inversion geometry and related topics, including complex chains, Chapter VII with affine and euclidean geometries of three dimensions and Chapter VIII with non-euclidean geometries.

Chapter IX is concerned with theorems on sense and separation, a large body of such theorems being proved on the basis of A, E, S and P without use of C and R. Sense classes of various sorts are defined in terms of elementary transformations. In euclidean space of two dimensions for an ordered set of three non-collinear points an elementary transformation is defined as the operation of replacing one of the three points in question by a point which is joined to it by a segment not meeting the line on the other two. In § 181 it is stated that the notion of right and left-handedness can be extended to curves by a limiting process. For a treatment of sense on curves in two dimensions without the use of such a limiting process reference may be made to an article by J. R. Kline.†

* Cf. page 96. Also footnote reference to Wilson and Lewis.

† J. R. Kline, "A definition of sense on closed curves in non-metrical analysis situs," *Annals of Mathematics*, vol. 19 (1918), pp. 185–200.

The reviewer feels that the volume under review is a valuable addition to the as yet rather restricted list of advanced mathematical treatises of high grade published in America.

R. L. MOORE.

NOTES.

AT a special meeting on April 23, 1920, the Council of the American Mathematical Society approved the formation of an American section of the international mathematical union and authorized its committee on the union to take the necessary steps to organize the section. The Council also adopted a resolution that the publication of a journal of mathematical abstracts is very desirable, and authorized its committee on bibliography to take steps toward securing the financial support necessary for such a journal. It was agreed that the representatives of the Society in the division of physical sciences of the National research council should present these projects before the division. Accordingly, at its meetings on April 28-29, 1920, the division adopted resolutions recommending to the National research council that the American section of mathematics organized under the auspices of the division be made the authorized agent of the council in the organization of the proposed international mathematical union, and its representative in that body when organized.

At the same meetings the Mathematical association of America was given the right to nominate one member of the division. The number of members at large was increased by one and Professor G. D. BIRKHOFF was elected as the additional member. Professor OSWALD VEBLEN was elected a member of the executive committee of the division.

The project for the publication of a journal of mathematical abstracts was approved and a committee consisting of Professors L. E. DICKSON, OSWALD VEBLEN, and H. S. WHITE was appointed to work out details and consult with the finance committee of the council as to securing the necessary funds.

A committee was also appointed to secure a revolving fund for the publication of important scientific books and papers commercially unattractive to regular publishing houses. Provision was made for the appointment of research committees

on algebraic numbers and statistics. The division approved a plan for establishing research fellowships in mathematics, to be administered by the division.

THE April number (volume 21, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Differential equations containing arbitrary functions," by G. A. BLISS; "Functions of lines in ballistics," by G. A. BLISS; "On the summability of the developments in Bessel's functions," by C. N. MOORE; "One-parameter families and nets of ruled surfaces and a new theory of congruences," by E. J. WILCZYNSKI; "Nets of space curves," by G. M. GREEN; "A set of postulates for fields," by N. WIENER; "A theorem on modular covariants," by O. C. HAZLETT.

THE March number (volume 21, number 3) of the *Annals of Mathematics* contains: "A Green's theorem in terms of Lebesgue integrals," by H. E. BRAY; "Bilinear operations generating all operations in a rational domain Ω," by NORBERT WIENER; "On the enumeration of proper and improper representations in homogeneous forms," by E. T. BELL; "A proof of Jordan's theorem about a simple closed curve," by J. W. ALEXANDER; "Linear order in three dimensional euclidean and double elliptic spaces," by G. H. HALLETT, JR.; "Further properties of the general integral," by P. J. DANIELL; "Summability of double series," by L. L. SMAIL; "The fundamental theorem of celestial mechanics," by J. L. COOLIDGE,

AT the meeting of the National Academy of Sciences held at Washington, April 27, the following mathematical papers were read: By Professor L. E. DICKSON, "Recent notable progress in the theory of numbers"; by Professor EDWARD KASNER, "Geodesics and relativity." Professor H. F. BLICHFELDT, of Stanford University, has been elected a member of the academy, and Professor CAMILLE JORDAN, of the Collège de France, a foreign associate.

THE following university courses in mathematics are announced:

COLUMBIA UNIVERSITY. The list of courses announced for

the academic year 1920–1921 in the May BULLETIN should, have included the following:—By Professor C. J. KEYSER: Philosophy of mathematics, four hours.

CORNELL UNIVERSITY (academic year 1920–1921).—By Professor J. H. TANNER: Mathematics of finance, two hours.—By Professor VIRGIL SNYDER: Algebraic geometry, three hours; Symposium of mathematics, two hours.—By Professor F. R. SHARPE: Fourier series and the potential theory, three hours (first term).—By Professor W. B. CARVER: Metric geometry, three hours.—By Professor ARTHUR RANUM: Theory of numbers, three hours (second term).—By Professor D. C. GILLESPIE: Differential equations, three hours.—By Professor W. A. HURWITZ: Differential equations of mathematical physics, three hours.—By Professor C. F. CRAIG: Theory of functions of a complex variable, three hours.—By Professor F. W. OWENS: Advanced calculus, three hours.—By Dr. F. W. REED: Celestial mechanics, three hours.—By Dr. H. B. OWENS: Projective geometry, three hours.—By Dr. G. M. ROBISON: Infinite series, three hours.—By Mr. H. S. VANDIVER: Modern algebra and theory of equations, three hours.—By Dr. H. C. M. MORSE: Differential geometry, three hours.

HARVARD UNIVERSITY (academic year 1920–1921).—All courses meet three times a week throughout the year, except those marked *, which meet for half a year.—By Professor W. F. OSGOOD: Dynamics (second course); The theory of functions (second course, part I): functions on an algebraic configuration*.—By Professor C. L. BOUTON: Introduction to modern geometry and algebra; Geometrical transformations, with special reference to the work of Sophus Lie.—By Professor C. L. COOLIDGE: Probability*; Properties of polynomials and invariants*; Line geometry.—By Professor E. V. HUNTINGTON: The fundamental concepts of mathematics*.—By Professor O. D. KELLOGG: Introduction to the theory of potential functions and Laplace's equation*.—By Professor D. R. CURTISS (of Northwestern University): The analytical theory of heat and problems in elastic vibrations*; The theory of functions (second course, part II): functions defined by linear differential equations of the second order*.—By Professor G. D. BIRKHOFF: Differential and integral calculus

(advanced course); Developments in series*; Difference equations*.—By Professor W. C. GRAUSTEIN: The theory of functions (introductory course); Differential geometry of curves and surfaces.

Professors KELLOGG and BIRKHOFF will conduct a fortnightly seminar in analysis. Courses of research are also offered by Professor OSGOOD in the theory of functions, by Professor BOUTON in the theory of point transformations, by Professor COOLIDGE in geometry, by Professor KELLOGG in the theory of potential functions, by Professor BIRKHOFF in the theory of differential equations, and by Professor GRAUSTEIN in geometry.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY (summer session).—By Professor F. S. WOODS: Analytic geometry.—By Mr. L. H. RICE: Elements of differential equations.—By Professors C. L. E. MOORE, H. B. PHILLIPS, F. L. HITCHCOCK, and DOUGLASS; Theoretical mechanics (introductory course).

UNIVERSITY OF CALIFORNIA (academic year 1920–1921).—By Professor M. W. HASKELL: Higher plane curves, three hours.—By Professor D. N. LEHMER: Geometry of four dimensions, three hours (first term).—By Professor FLORIAN CAJORI: History of mathematics, two hours; Teachers' course, three hours (first term).—By Professor T. M. PUTNAM: Partial differential equations, three hours (first term); Special analytic functions, three hours (second term).—By Professor W. D. MCDONALD: Functions of a complex variable, three hours.—By Professor B. A. BERNSTEIN: Logic of mathematics, two hours.—By Professor FRANK IRWIN: Introduction to higher algebra, three hours.—By Dr. PAULINE SPERRY: Differential geometry, three hours (second term).

UNIVERSITY OF WISCONSIN (summer session).—By Professor E. P. LANE: Differential equations, five hours; Modern analytic geometry, three hours.—By Professor H. W. MARCH: Mechanics, five hours; Differential equations of mathematical physics, five hours.—By Professor E. B. SKINNER: Differential geometry, five hours; Special topics in algebra, three hours.—By Professor ARNOLD DRESDEN: Elliptic integrals, five hours; Theory of point sets, five hours.

UNIVERSITY OF STRASBOURG.—The reorganized University of Strasbourg announces that it is now in a position to offer a complete course of research for the doctorate in mathematics and mathematical physics, as well as the usual introductory courses in analysis, mechanics and astronomy. The following research courses will be given during the academic year 1920–1921: First semester (November 1, 1920–February 28, 1921).—By Professor BAUER: Quanta theory and the structure of atoms, three hours.—By Professor FRÉCHET: Theory of chance, two hours; Integral equations, one hour. Second semester (March 1, 1921–June 30, 1921).—By Professor BAUER: Statistical applications of quanta theory, three hours. —By Professor FRÉCHET: Applications of the theory of chance, one hour; Functions of lines, two hours.—By Professor VILLAT: The motion of a solid in a viscous fluid, two hours.—By Professor PÉRÈS: Transformations of surfaces applicable to quadrics, two hours.—By Professor VALIRON: Dirichlet's series and factorial series, two hours.

Further information with regard to mathematics courses may be obtained from M. le Directeur de l'Institut de Mathématiques de l'Université, Strasbourg, Bas-Rhin, France, and details concerning lodgings, etc., and courses in the French language for foreign students given during the summer of 1920 by the Faculty of letters, from the Comité de Patronage des Etudiants étrangers, Université, Strasbourg.

PROFESSOR PAUL APPELL, honorary dean of the faculty of sciences of the University of Paris, has been made rector of the Paris academy, as successor to the late LUCIEN POINCARÉ.

SIR THOMAS MUIR has recently presented his collection of about 2,500 mathematical books and pamphlets to the South African Public Library at Capetown. It consists largely of sets of periodicals, some of them now very rare, and of a special library on the theory of determinants and allied subjects that is probably the most complete in existence.

PROFESSOR D. R. CURTISS, of Northwestern University, has been granted leave of absence to accept the position of lecturer in mathematics at Harvard University during the coming academic half-year.

PROFESSOR L. E. DICKSON, of the University of Chicago, has been elected a member of the American Philosophical Society.

ASSISTANT professor S. LEFSCHETZ, of the University of Kansas, has been promoted to an associate professorship of mathematics.

AT the University of Oklahoma, associate professor E. P. R. DUVAL has resigned, and Mr. E. E. COWAN has been appointed instructor in mathematics.

AT Adelbert College, Western Reserve University, Dr. W. G. SIMON has been promoted to an assistant professorship of mathematics. Dr. C. A. NELSON, of the University of Kansas, has been appointed instructor in mathematics.

ASSISTANT professor J. V. MCKELVEY, of Iowa State College, has been promoted to an associate professorship of mathematics.

DR. I. A. BARNETT, Benjamin Peirce instructor at Harvard University during the academic year 1919–1920, and Mr. H. R. BRAHANA, of Princeton University, have been appointed instructors in mathematics at the University of Illinois.

DR. C. N. REYNOLDS, of Wesleyan University, has been appointed instructor in mathematics at Dartmouth College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CASTELNUOVO (G.). Lezioni di geometria analitica. Puntata 1. 4a edizione. Milano-Roma-Napoli, soc. ed. Dante Alighieri, di Albrighi, Segati e C. (Città di Castello, S. Lapi), 1919. 8vo. 8 + 320 pp. L. 30.00

FOWLER (R. H.). The elementary differential geometry of plane curves. (Cambridge Mathematical Tracts, No. 20.) Cambridge, University Press, 1920. Demy 8vo. 7 + 105 pp. 6s.

GANGULI (S.). Lectures on the theory of plane curves delivered to post-graduate students in the University of Calcutta. Parts 1 and 2. Calcutta, University of Calcutta, 1919. 10 + 140 + 14 + 212 pp. + 16 plates.

SHUGERT (S. P.). The resolvents of König and other types of symmetric functions. (Diss., University of Pennsylvania.) Lancaster, Pa., 1919. 19 pp.

ZOLESE (C.). Matematica pura o matematica applicata? Discorso, il 30 aprile, 1919. Città di Castello, tip. Grifani-Donati, 1919. 8vo. 21 pp.

II. ELEMENTARY MATHEMATICS.

ASCOLI (G.). Complementi di geometria per gli istituti tecnici. 4a edizione. Livorno, Giusti, 1920. 8vo. 10 + 237 pp. L. 5.40

BORTOLOTTI (E.). Aritmetica generale ed algebra per i licei classici e moderni. Volume 1, per la prima classe. Milano-Roma-Napoli, soc. ed. Dante Alighieri, di Albrighi, Segati e C. (Città di Castello, S. Lapi), 1919. 8vo. 197 pp. L. 3.00

BURALI-FORTI (C.). See PENSA (A.).

BURALI-FORTI (C.) e RAMORINO (A.). Elementi di algebra per le scuole medie inferiori. 6a edizione. Torino, casa ed. G. B. Petrini, di G. Gallizio (V. Bona), 1920. 8vo. 7 + 142 pp. L. 3.00

CATANIA (S.). Elementi di aritmetica ed algebra per le scuole secondarie inferiori. 7a edizione, migliorata e semplificata, e adattata pure ai nuovi proposti programmi. Catania, N. Giannotta (V. Giannotta), 1919. 16mo. 8 + 382 pp. L. 4.20

JENKINS (F.). See LENNES (N. J.).

LENNES (N. J.) and JENKINS (F.). Applied arithmetic, the three essentials. Books 2 and 3. (Lippincott's School Text Series.) Philadelphia, Lippincott, 1920. 9 + 294 + 9 + 340 pp. $0.80 + 0.88

PALATINI (F.). Algebra, ad uso delle scuole medie superiori. Parte 1-2. 5a edizione. Torino, casa ed. G. B. Petrini, di G. Gallizio (tip. Elzeviriana, Olivero e C.), 1920. 8vo. 76 + 126 pp. L. 6.50

PEANO (G.). See TAVOLE.

PENSA (A.). Elementi di geometria ad uso delle scuole secondarie inferiori. Prefazione di C. Burali-Forti. 6a edizione. Torino, casa ed. G. B. Petrini, di G. Gallizio (V. Bona), 1919. 8vo. 11 + 198 pp. con due prospetti. L. 4.00

RAMORINO (A.). See BURALI-FORTI (C.).

REEVE (W. B.). See SCHORLING (R.).

SCHORLING (R.) and REEVE (W. B.). General mathematics. Boston, Ginn, 1919. 16 + 488 pp. $1.48

STRAZZERI (V.). Geometria elementare. Palermo, D. Capozzi, 1919. 8vo. 8 + 272 pp. L. 5.00

TAVOLE numeriche (quadrato, cubo, radice quadrata e cubica, logaritmo, reciproco, seno, coseno, tangente, cotangente, logaritmi naturali, tavole d'interesse, ecc.) con prefazione e note di Giuseppe Peano. Torino, Unione tipografico-editrice, 1920. 16mo. 35 pp.

TESTI (G. M.). Corso di matematiche, ad uso delle scuole secondarie superiori e più specialmente degli istituti tecnici. Volume 5: Complementi d'algebra. 3a edizione, corretta ed aumentata, coll'aggiunta di un'appendice sulla funzione derivata e sugli integrali. Livorno, Giusti, 1920. 8vo. 9 + 330 pp. L. 5.00

——. Elementi d'algebra, ad uso più specialmente dei licenziandi delle scuole tecniche. 16a edizione riveduta. Livorno, Giusti, 1920. 16mo. 6 + 86 pp. L. 1.80

——. Elementi di geometria, ad uso degli alunni delle scuole tecniche. 23a edizione riveduta. Livorno, Giusti, 1920. 16mo. 11 + 239 pp. L. 4.00

——. Elementi di matematica, ad uso degli alunni delle scuole normali. Fasc. 3, per la 2a classe. 6a edizione riveduta. Livorno, Giusti, 1920. 16mo. 8 + 107 pp. L. 2.00

III. APPLIED MATHEMATICS.

Brahe (T.). Tychonis Brahe Dani opera omnia. Edidit J. L. E. Dreyer. Tomus 6. Hauniæ, Libreria Gyldendaliana, 1920. 5 + 375 pp.

Brose (H. L.). See Freundlich (E.).

Deckert (A.). Mechanik. Kempten, 1919.

Delsol (E.). Considérations sur les unités fondamentales de la physique. Paris, chez l'auteur, 1919. 8vo. 31 pp.

Dorgeot (E.). Cinématique théorique et appliquée. Paris, Dunod et Pinat, 1919. 4to. 272 pp. Fr. 54.00

Dreyer (J. L. E.). See Brahe (T.).

Einstein (A.). See Freundlich (E.).

File (L. K.). See Moir (H.).

Freundlich (E.). The foundations of Einstein's theory of gravitation. Authorized English translation by H. L. Brose. With a preface by A. Einstein and an introduction by H. H. Turner. Cambridge, University Press, 1920. Demy 8vo. 16 + 61 pp. 5s.

Garneri (A.). Corso elementare di disegno geometrico, ad uso delle scuole medie, complementari, professionali, ecc. Parte 2: Proiezioni ortogonali con applicazioni alla meccanica e alla teoria delle ombre prospettiva parallela. 9a edizione, riveduta, corretta, aumentata. Torino, ditta G. B. Paravia e C. (Rocca S. Casciano, L. Capelli), 1919. 16mo. 8 + 180 pp. L. 3.00

Gibbs (W.). L'équilibre des substances hétérogènes. Exposé abrégé, traduit et complété de notes explicatives par Georges Matisse. Paris, Gauthier-Villars, 1919. 16mo. 8 + 102 pp. Fr. 5.25

Goffi (C.). L'apprendista meccanico. 2a edizione, riveduta et ampliata. (Manuali Hoepli.) Milano, Hoepli, 1920. 24mo. 19 + 381 pp. L. 7.50

Jung (H.). See Nölke (F.).

Matisse (G.). See Gibbs (W.).

Moir (H.), Craig (J. D.), File (L. K.), Maclean (A. T.) and Wolfenden (H. H.). Sources and characteristics of the principal mortality tables. (Actuarial Studies, No. 1.) New York, Actuarial Society of America, 1919. 4 + 79 pp. $1.25

Turner (H. H.). See Freundlich (E.).

Wilson (E. B.). Aeronautics. A class text. New York, Wiley, 1920. 8 + 265 pp. $4.00